# GENERALIZED ADJOINT ACTIONS

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Abstract. The aim of this paper is to generalize the classical formula $e^x y e^{-x} = \sum\limits_{k\ge 0} \frac{1}{k!}(ad\ x)^k(y)$ by replacing $e^x$ with any formal power series $f(x) = 1 + \sum\limits_{k\ge 1} a_k x^k$. We also obtain combinatorial applications to $q$-exponentials, $q$-binomials, and Hall-Littlewood polynomials.

## 1. Notation and main results

One of the most fundamental tools in Lie theory, the adjoint action of Lie groups on their Lie algebras, is based on the following formula:

$$e^x y e^{-x} = e^{ad\ x}(y) = \sum_{k\ge 0} \frac{1}{k!}(ad\ x)^k(y)\ , \tag{1.1}$$

where $(ad\ x)^k(y) = [x, [x, \dots, [x, y], \dots]]$ and $[a, b] = ab - ba$.

The aim of this paper is to generalize (1.1) by replacing $e^t$ with any formal power series

$$f = f(t) = 1 + \sum_{k\ge 1} a_k t^k \tag{1.2}$$

over a field $\mathbb{k}$.

For any formal power series (1.2) over $\mathbb{k}$ define polynomials

$$P_k(t) = P_{f,k}(t) = (-1)^k \det \begin{pmatrix} 1 & a_1 t & a_2 t^2 & \dots & a_k t^k \\ 1 & a_1 & a_2 & \dots & a_k \\ 0 & 1 & a_1 & \dots & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

for $k = 0, 1, 2, \dots$. (with the convention that $P_0(t) = 1$). Clearly, $P_k(1) = 0$ for $k \ge 1$. Using Cramer's rule with respect to the last column, one obtains a recursion $P_k(t) = a_k t^k - \sum\limits_{i=1}^{k} a_i P_{k-i}(t)$.

The following result is, probably, well-known (for readers' convenience, we prove it in Section 2).

**Theorem 1.1.** *For any power series $f(t)$ as in (1.2), one has*

$$\frac{f(tx)}{f(x)} = \sum_{k\ge 0} P_{f,k}(t) \cdot x^k \tag{1.3}$$

*and*

$$P_{f,k}(st) = \sum_{i=0}^{k} P_{f,i}(s) P_{f,k-i}(t) t^i \tag{1.4}$$

*for all $k \ge 0$.*

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Furthermore, for any algebra $\mathcal{A}$ over $\Bbbk$, a subset $\mathbf{q}=\{q_1,\ldots,q_k\}\subset\Bbbk$, $x,y\in\mathcal{A}$, and $k\geq 1$ define

$$(ad\ x)^{\mathbf{q}}(y)=[x,[x,\ldots,[x,y]_{q_1},\ldots]_{q_{k-1}}]_{q_k}$$

where $[a,b]_q:=ab-qba$. It is easy to see that

$$(ad\ x)^{\mathbf{q}}(y)=\sum_{j=0}^{k}(-1)^j e_j(q_1,\ldots,q_k)\cdot x^{k-j}yx^j\ , \tag{1.5}$$

where $e_j(q_1,\ldots,q_k)$ is the $j$-th elementary symmetric function.

**Theorem 1.2.** *Let $\mathcal{A}$ be $\Bbbk$-algebra and suppose that $f$ is any power series* (1.2) *with $a_k\neq 0$ for $k\geq 1$. Then*

$$f(x)yf(x)^{-1}=y+\sum_{k\geq 1}a_k(ad\ x)^{\mathbf{q}_k}(y)\ , \tag{1.6}$$

*for any $x,y\in\mathcal{A}$, where $\mathbf{q}_k=\{q_{1k},\ldots,q_{kk}\}$ is the set of roots of $P_{f,k}(t)$.*

**Remark 1.3.** A formula for $f(x)yf(x)^{-1}$ without assumption that all $a_k\neq 0$ is given in Proposition 2.3.

**Remark 1.4.** Strictly speaking, the formula (1.6), similarly to (1.1) requires a completion of $\mathcal{A}$. One can bypass this by replacing $x$ with $\tau\cdot x$ where $\tau$ is a purely transcendental element of $\Bbbk$ so that the right hand side of (1.6) becomes a power series in $\tau$ (and, maybe extending $\Bbbk$ if it lack such an element).

**Remark 1.5.** The subsets $\mathbf{q}_k$ may belong to an extension of $\Bbbk$, however, the operators $(ad\ x)^{\mathbf{q}_k}$ are defined over $\Bbbk$ due to (1.5) because all symmetric functions in $\mathbf{q}_k$ belong to $\Bbbk$.

It is easy to see that if $a_k=\frac{1}{k!}$ for all $k$, then $P_k(t)=\frac{(t-1)^k}{k!}$ which immediately recovers (1.1). Suppose now that $a_k=\frac{1}{[k]_q!}$ for all $k$, where $k_q!=[1]_q\cdots[k]_q$ is the $q$-factorial and $[\ell]_q=1+q+\cdots+q^{\ell-1}$. We will show (Proposition 2.5) that $P_{f,k}(t)=\dfrac{(t-1)(t-q)\cdots(t-q^{k-1})}{[k]_q!}$ for $f(t)=e_q^t=\sum\limits_{k\geq 0}\dfrac{t^k}{[k]_q!}$, therefore, recover the following famous result (see e.g., [3]).

**Theorem 1.6.** *Let $e_q^x=\sum\limits_{k\geq 0}\dfrac{x^k}{[k]_q!}$ be the $q$-exponential. Then $e_q^x\cdot y\cdot(e_q^x)^{-1}=\sum\limits_{k\geq 0}\dfrac{1}{[k]_q!}(ad\ x)^{\{1,q,\ldots,q^{k-1}\}}(y)$.*

On the other hand, combining Theorem 1.1 and Proposition 2.5, we recover the following well-known properties of $q$-exponentials and $q$-binomials:

$$e_q^{q^nx}=e_q^x\left(1+\sum_{k=1}^{n}\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{[k]_q!}x^k\right)$$

for $n\geq 0$, in particular,

$$e_q^{qx}=e_q^x\cdot(1+(q-1)x)$$

and

$$1+\sum_{k=1}^{n}\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{[k]_q!}x^k=\prod_{i=1}^{n}(1+(q-1)q^{i-1}x)\ .$$

We conclude with a curious observation that the polynomials $P_{f,k}(t)$ are related to the Hall-Littlewood symmetric polynomials.

**Proposition 1.7.** *Suppose that $f(t)=\prod\limits_{k\geq 1}(1-x_kt)$. Then*

$$P_{f,k}(t)=Q_{(k)}(\mathbf{x};t)$$

*for all $k \geq 0$, where $\mathbf{x} = \{x_k, k \geq 0\}$ is viewed as an infinite set of variables, $Q_\lambda(\mathbf{x};t)$ is Hall-Littlewood polynomial (*[2, Section 3.2]*), and $(k)$ is a one-row Young diagram with $k$ cells. In particular,*

$$Q_{(k)}(\mathbf{x};t) = (-1)^k \det \begin{pmatrix} 1 & -e_1 t & e_2 t^2 & \dots & (-1)^k e_k t^k \\ 1 & -e_1 & e_2 & \dots & (-1)^k e_k \\ 0 & 1 & -e_1 & \dots & (-1)^{k-1} e_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & -e_1 \end{pmatrix}$$

*for all $k \geq 0$, where $e_k = e_k(\mathbf{x})$ is the $k$-th elementary symmetric function.*

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## 2. Proofs

**Proof of Theorem 1.1**. We need the following well-known fact (attributed to Wronski, see e.g., [1]).

**Lemma 2.1.** *Let $f$ be any formal power series* (1.2)*. Then $\dfrac{1}{f(t)} = 1 + \sum_{k\geq 1} D_k(f) t^k$, where*

$$D_k(f) = (-1)^k \det \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_k \\ 1 & a_1 & a_2 & \dots & a_{k-1} \\ 0 & 1 & a_1 & \dots & a_{k-2} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

*(with the convention $D_0(f) = 1$).*

The following generalization of Lemma 2.1 is, apparently, well-known (for readers' convenience we prove it here).

**Lemma 2.2.** *Let $f(t) = 1 + \sum\limits_{k\geq 1} a_k t^k$, $g(t) = 1 + \sum\limits_{k\geq 1} b_k t^k$ be formal power series. Then*

$$\frac{g(t)}{f(t)} = \sum_{k\geq 0} D_k(g,f) t^k \ ,$$

*where* $D_k(g,f) = (-1)^k \det \begin{pmatrix} 1 & b_1 & \dots & b_{k-1} & b_k \\ 1 & a_1 & \dots & a_{k-1} & a_k \\ 0 & 1 & \dots & a_{k-2} & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$ *(with the convention $D_0(g,f) = 1$).*

**Proof.** Indeed, using Lemma 2.1, we obtain (with the convention $b_0 = 1$):

$$\frac{g(t)}{f(t)} = g(t) \cdot \frac{1}{f(t)} = \left(\sum_{i\geq 0} b_i t^i\right) \left(\sum_{j\geq 0} D_j(f) t^j\right) = \sum_{k\geq 0} d_k t^k$$

where

$$d_k = \sum_{i=0}^{k} b_i D_{k-i}(f) = (-1)^k \det \begin{pmatrix} b_0 & b_1 & \dots & b_{k-1} & b_k \\ 1 & a_1 & \dots & a_{k-1} & a_k \\ 0 & 1 & \dots & a_{k-2} & a_{k-1} \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

by Cramer rule because $b_iD_{k-i}(f)=(-1)^k\det\begin{pmatrix}0&0&\dots&b_i&\dots&0\\1&a_1&\dots&a_i&\dots&a_k\\0&1&\dots&a_{i+1}&\dots&a_{k-1}\\\vdots&\vdots&\ddots&\ddots&\ddots&\ddots\\0&0&\dots&1&a_1&a_2\\0&0&\dots&\dots&1&a_1\end{pmatrix}$.

The lemma is proved. $\square$

Then taking $b_k=a_kt^k$ for $k\ge 1$ in Lemma 2.2, we obtain (1.3).

To prove (1.4), compute $\frac{f(stx)}{f(x)}$ in two ways, using the first assertion:

$$\frac{f(stx)}{f(x)}=\sum_{k\ge 0}P_{f,k}(st)\cdot x^k$$

and $\frac{f(stx)}{f(x)}=\frac{f(stx)}{f(tx)}\cdot\frac{f(tx)}{f(x)}=\left(\sum_{i\ge 0}P_{f,i}(s)\cdot(tx)^i\right)\left(\sum_{j\ge 0}P_{f,j}(t)\cdot x^j\right)$. Comparing the coefficients of $x^k$ in both series, we obtain (1.4).

Theorem 1.1 is proved. $\square$

**Proof of Theorem 1.2**. We need the following result.

**Proposition 2.3.** *For any power series $f$ as in* (1.2) *one has:*

*(a) $P_{f,k}(t)=\sum\limits_{j=0}^{k}a_{k-j}D_j(f)\cdot t^j$ for all $k\ge 0$.*

*(b) $f(x)yf(x)^{-1}=y+z_1+z_2+\cdots$, where $z_k=\sum\limits_{i=0}^{k}a_iD_{k-i}(f)\cdot x^iyx^{k-i}$ for all $k\ge 1$.*

**Proof.** Prove (a). Indeed, using Lemma 2.1, we obtain:

$$\frac{f(tx)}{f(x)}=\sum_{i,j\ge 0}\left(a_it^ix^i\right)\left(D_j(f)x^j\right)=\sum_{k\ge 0}\left(\sum_{i=0}^{k}a_iD_{k-i}(f)\cdot t^i\right).$$

This together with Theorem 1.1 proves (a).

Prove (b) now. Indeed,

$$f(x)yf(x)^{-1}=\sum_{i,j\ge 0}\left(a_ix^i\right)y\left(D_j(f)x^j\right)=\sum_{k\ge 0}\left(\sum_{j=0}^{k}a_iD_{k-i}(f)\cdot x^iyx^{k-i}\right)$$

This proves (b). $\square$

Now we can finish the proof of Theorem 1.2. Indeed, suppose that $P_{f,k}(t)$ is factored as

$$P_{f,k}(t)=a_k(t-q_{1k})\cdots(t-q_{kk})\ .$$

Then, by Proposition 2.3(a), $a_iD_{k-i}=a_k(-1)^{k-i}e_{k-i}(q_{1k},\dots,q_{kk})$ for $i=0,\dots,k$. Therefore, in the notation of Proposition 2.3(b),

$$z_k=\sum_{i=0}^{k}a_k(-1)^{k-i}e_{k-i}(q_{1k},\dots,q_{kk})\cdot x^iyx^{k-i}=a_k(ad\ x)^{\mathbf{q}_k}(y)$$

for all $k\ge 1$, which together with Proposition 2.3(b) verifies (1.6).

Theorem 1.2 is proved. $\square$

**Proposition 2.4.** $P_{e_q^t,k}(t)=\dfrac{(t-1)(t-q)\cdots(t-q^{k-1})}{[k]_q!}$ *for all $k\ge 1$.*

**Proof.** It suffices to show that $P_{e_q^t,k}(q^a) = 0$ for all $0 \le a < k$. We proceed by induction in such pairs $(a, k)$. If $a = 0$, then we have nothing to prove since $P_{f,k}(1) = 0$ for all $f$.

Using Theorem 1.1, we obtain:

$$P_{f,k}(q^a) = \sum_{i=0}^{k} P_{f,i}(q^b) P_{f,k-i}(q^{a-b}) q^{(a-b)i} \ .$$

Taking $f = e_q^t$, $1 \le b \le a < k$, and using the inductive hypothesis, this gives $P_{f,k}(q^a) = 0$ for any $1 \le a < k$.

The proposition is proved. $\square$

**Corollary 2.5.** *For all $k \geq 1$ one has:* $\det \begin{pmatrix} 1 & \frac{t}{[1]_q} & \frac{t^2}{[2]_q} & \cdots & \frac{t^k}{[k]_q} \\ 1 & \frac{1}{[1]_q} & \frac{1}{[2]_q} & \cdots & \frac{1}{[k]_q} \\ 0 & 1 & \frac{1}{[1]_q} & \cdots & \frac{1}{[k-1]_q} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 & \frac{1}{[1]_q} \end{pmatrix} = \dfrac{(1-t)(q-t)\cdots(q^{k-1}-t)}{[k]_q!}.$

**Proof of Proposition 1.7**. Indeed, if $f(t)$ is as in Proposition 1.7, then

$$\frac{f(tu)}{f(u)} = \prod_{k \geq 1} \frac{1 - x_k t u}{1 - x_k u} = \sum_{k \geq 0} Q_{(k)}(\mathbf{x}; t) u^k$$

by [2, Equations (2.10) and (2.13)]. This and Theorem 1.1 imply that $P_{f,k} = Q_{(k)}(\mathbf{x}; t)$ for all $k \geq 0$, which proves the first assertion of Proposition 1.7.

To prove the second assertion, note that $a_k = (-1)^k e_k(\mathbf{x})$ for all $k \geq 0$ because of the well-known formula (see e.g., [2, Section 1.2]):

$$\prod_{k \geq 1} (1 - x_k t) = \sum_{k \geq 0} (-1)^k e_k(\mathbf{x}) t^k \ .$$

This and the first assertion of Proposition 1.7 imply the second assertion of the proposition. $\square$

## References


[1] A. Inselberg, On determinants of Toeplitz-Hessenberg matrices arising in power series, *J. Math. Anal. Appl.* **63** (1978), no. 2, 347–353.

[2] I. Macdonald, *Symmetric functions and Hall polynomials* Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[3] A. Volkov, Beyond the "pentagon identity," *Lett. Math. Phys.* **39** (1997), no. 4, 393–397.



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